

ON REAL INTRINSIC WALL CROSSINGS

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We study moduli space stabilization of a class of BPS configurations from the perspective of the real intrinsic Riemannian geometry. Our analysis exhibits a set of implications towards the stability of the D-term potentials, defined for a set of abelian scalar fields. In particular, we show that the nature of marginal and threshold walls of stabilities may be investigated by real geometric methods. Interestingly, we find that the leading order contributions may easily be accomplished by translations of the Fayet parameter. Specifically, we notice that the various possible linear, planar, hyper-planar and the entire moduli space stabilities may easily be reduced to certain polynomials in the Fayet parameter. For a set of finitely many real scalar fields, it may be further inferred that the intrinsic scalar curvature defines the global nature and range of vacuum correlations. Whereas, the underlying moduli space configuration corresponds to a non-interacting basis at the zeros of the scalar curvature, where the scalar fields become un-correlated. The divergences of the scalar curvature provide possible phase structures, viz., wall of stability, phase transition, if any, in the chosen moduli configuration. The present analysis opens up a new avenue towards the stabilizations of gauge and string moduli.

Keywords: Wall Crossings, Intrinsic Geometry, Supersymmetric Configurations, Moduli Stabilization, D-terms, Fayet Models.

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1 Introduction

The moduli Space geometry is a cornerstone of modern string theory compactifications and divulges the nature of a class of intriguing quantum field theory, supersymmetric gauge theory configurations and string theory dynamics in all possible dimensions [1, 2]. In fact, it has been a long standing problem how to resolve the fundamental issue of moduli stabilization in cosmology, gauge theory and superstring theory [3–13]. Moreover, the problem gets a more interesting facet from the perspective of moduli stabilization and flux compactifications [14–16], which are a class of configurations involving the D-terms contributions. Nevertheless, our analysis leads to an interesting conclusion for the notion of the moduli stability of underlying configurations. Such an introduction of the Fayet model [17, 18] captures the basic leading order features of the moduli stabilization problem. The promising application of the present analysis concerns certain decaying supergravity configurations. Herewith, we shall consider the intrinsic real geometric framework to analyze the phenomena of wall crossings, and in particular, we shall focus our attention on the D -term potentials.

It is worth mentioning that the charged extremal black holes in $D = 4$, $N = 2$ supergravity may be characterized by a set of electric-magnetic charges $\{q_J, p^I\}$, which arise from the flux integrals of the field strength tensor and the corresponding Poincaré dual. On the other hand, the scalar fields arising from the compactification of both the string theory and M-theory yield a set of moduli fields, which in effect parameterize a certain compact internal space. The extremal charged black hole solutions can be viewed as BPS solitons, which interpolate between the asymptotic infinity and $AdS_2 \times S^2$ near horizon geometry [19, 20]. The spherical symmetry, in turn, determines the underlying interpolation as the radial evolution of the scalar moduli, which encodes the consequent changes of the underlying compact internal manifold. Moreover, one finds that there exists the flat Minkowskian manifold at the asymptotic infinity, for a given scalar moduli configuration. The asymptotic ADM mass, as an arbitrary parameter, is described by the complex central extension Z_∞ of the underlying supersymmetry algebra [19, 20]. Subsequently, it turns out that the ADM mass satisfies $M(p, q, \varphi^a) = |Z_\infty|$.

In such cases, the near horizon geometry of an arbitrary extremal black hole reduces to the $AdS_2 \times S^2$ manifold, which describes the underlying Bertotti-Robinson vacuum. The area of the black hole horizon A , and hence its macroscopic entropy is given as $S_{macro} = \pi|Z_\infty|^2$. However, it turns out that the radial variation of the moduli is described by a damped geodesic equation, which flows to an attractive fixed point at the horizon. Thus, it may solely be determined by the charges

carried out by the chosen extremal black hole. Such attractors [21] have been further studied from the perspective of the criticality of the black hole effective potential in $D = 4$, $N = 2$ supergravities coupled to n_V abelian vector multiplets. An asymptotically flat extremal black hole background may in turn be described by $(2n_V + 2)$ -dyonic charges and n_V -complex scalar fields. Thus, the scalar moduli parameterize a n_V -dimensional special Kähler manifold, see [21] and references therein.

From the perspective of the attractor horizon geometries, the study of supergravity theories led to the fact that an extremal dyonic black hole has a non-vanishing Bekenstein-Hawking entropy, see for example [22]. In Ref. [22], such an analysis has been given for 1/2-BPS and non-BPS (non-supersymmetric) attractors with a non-vanishing central charge, in the context of $D = 4$, $\mathcal{N} = 2$ ungauged supergravity coupled to a n_V -number of abelian vector multiplets. Further, this leads to an interesting classification of the (i) orbits of the classical U-duality group, in the symplectic representation, and (ii) moduli spaces associated with the non-BPS attractors, in the context of symmetric special Kähler geometries. From the perspective of the present analysis, the case of non-extremal black brane configurations may be similarly considered by adding the corresponding anti-branes to the extremal black brane configurations. Thus, the computation of the associated black brane entropy may be performed in both the microscopic and the macroscopic descriptions, viz., [23, 24]. In fact, Refs. [23, 24] show that such a consideration leads to a perturbative matching between the S_{micro} and S_{macro} , for a set of brane charges and a given mass. Interestingly, the addition of the mass to an extremal configuration defines a nonextremal configuration.

From the perspective of thermodynamic geometry, it turns out that the problem involved could be examined at the attractor fixed point(s), where the moduli fields are stabilized via the attractor equations [25–35], and thus expressed in terms of the invariant charges of the configuration. The promising geometry, thus arising at an attractor fixed point, describes thermodynamic fluctuations in the entropy of the underlying black hole and a class of associated configurations [36–64]. Furthermore, some interesting directions have been explored for the non-extreme Calabi-Yau configurations [65–70]. In this direction, one may investigate the moduli stabilization problem further, from the perspective of [71–73]. In particular, one finds, for a given Kähler potential as a function of the moduli, that the case of the extreme Calabi-Yau manifolds corresponds to certain simplifications in the computation of the attractor flow equations. Interestingly, the intrinsic geometric transformations turn out to be extraordinarily informative, and in particular, such transformations uprise to certain symmetries of the effective potential of the theory. In such considerations, it is worth mentioning that the extremization problem involves an appropriate notion of the covariant derivative, and thus one needs to incorporate the contributions coming from the central charge of the theory [74–76], which offers appropriate setups for the general Kähler moduli configurations.

Nevertheless, the framework of the present paper outlines an appreciation of the extremization problem. Consequently, one may consider the examinations from the stability of the moduli configurations. For the string theory motivated backgrounds, such a system could be obtained by an appropriate compactification of a higher dimensional configuration. For a given finite dimensional moduli system, one may consider the Hessian of the effective potential and thereby employ the notion of the intrinsic geometry. In particular, it may be anticipated that such an investigation would correspond to a unified depiction of both the statistical fluctuations and intrinsic Riemannian geometric stabilities. Thus, it may be expected that there exists a certain geometric notion of the stability, for a given (black hole) effective potential. The classical configurations effectively reduce to an underlying thermodynamic system, at the attractor fixed point(s). From the viewpoint of the

present consideration, there exists a class of black hole solutions [77–82], leading to an interesting attractor flow and the corresponding thermodynamic configurations. Furthermore, there have been considerations of incorporating higher order corrections [83, 84], from the perspective of the string theory. In general, the notion of stability may be divulged from the perspective of the moduli flow. Here, we shall offer an associated examination from the viewpoint of the polynomial invariants.

The present investigation demonstrates that definite stability implications do arise from the intrinsic Riemannian geometry. For an extreme supergravity configuration, the stabilization problem finds a natural ground in the framework of the real intrinsic geometry. We stress that the number of harmonic modes defines underlying microscopic systems, which in the large charge limit, allow one to accurately configure the system under the inclusion of certain fluctuations. Whereas, the respective classical moduli systems may exactly be defined in terms of a set of scalar fields. In particular, it turns out that the fluctuation of the scalar potential, when it is considered as a function of the scalar fields, characterizes the chosen macroscopic attractor configuration. In general, we find that the underlying configuration, which possesses a prime importance towards the present interest, may be described by the D -term potential,

$$V_D(\varphi_i) = \left(\sum_{i \in \Lambda} q_i \varphi_i^\dagger \varphi_i + b \right)^2 \quad (1)$$

Here, the Fayet parameter b divulges the basic nature of the underlying supersymmetric theory and entails the decay property in the moduli configurations. At this juncture, it is worth mentioning that we wish to analyze the walls of stability of the BPS-configurations, and put our emphasis from the perspective of the intrinsic real Riemannian geometry. Henceforth, we shall consider $|q| = 1$. Such a choice follows from the fact that the moduli fields may be adjusted by an appropriate respective dilatation. Notice further that the marginal stability requires a set of uniformly distributed charges and thus the sign should be taken to be the same as that of the Fayet parameter. Whilst, the threshold stability requires an alternating set of charges. In this case, the relative sign of the vacuum value of constituent scalars is such that the effective potential vanishes for (i) equal values of the scalars and (ii) $b = 0$. This property follows from the fact that the effective potential is required to ensure the supersymmetry constraints at the vacuum. Herewith, we shall focus our attention on the moduli configurations, which are described by the single and two complex scalar fields. For such configurations, we shall analyze the walls of BPS stabilities from the perspective of the real intrinsic geometry.

The physical moduli may thus be described as an intrinsic fluctuating configuration, whose pair correlations satisfy a set of interesting planar and hyper-planar stability constraints. In order to appreciate a formal intrinsic picture, we shall consider a set of real scalar fields $\{x_i\}$ with an embedding $V : \{x_i\} \rightarrow R$, such that there exists a definite scalar potential for the real scalars. We shall intrinsically analyze the role of the possible limits: $b \mapsto -b$, $b \mapsto 0$ and $b \mapsto 1/b$, and thereby offer the intrinsic geometric criterion to the quantities. In particular, we shall exhibit how the planar and hyper-planar stability conditions vary under the reflection, translation and inversion of the Fayet parameter b . Notice that the reflection symmetry of the Fayet term is required, due to the supersymmetry constraints, and in fact only reflection symmetric combinations contribute to the physical nature of the underlying vacuum moduli configuration. In the subsequent section, a similar role of dilatation and restriction of the Fayet parameter is offered, from the perspective of the polynomial invariance.

An indispensable relevance of the concepts being divulged in this paper may thus be structured as a certain intriguing intrinsic real Riemannian geometry which arises from the nature of underlying marginal and threshold stable configurations.

Furthermore, it is interesting to note that the marginal stable configurations allow for BPS to non-BPS decays, whereas the corresponding threshold stable moduli configurations do not. Furthermore, we observe that the transformation $b \mapsto -b$ implies that the D-term threshold potentials satisfy $V_D^{2n}(x_i) = 0$, for all possible physical values of the Fayet parameter. In particular, it is worth to mention, for a positive value of the Fayet parameter and two real scalars $\{x_1, x_2\}$, that the threshold stable configurations require a preferred surface of constraint, e.g. $\langle x_1 \rangle_0 = 0$, $\langle x_2 \rangle_0 = b$, whereas the choice $\langle x_1 \rangle_0 = b$, $\langle x_2 \rangle_0 = 0$ leads to a negative value. In fact, the above cases only offer an illustration of the consideration, and indeed there exist a whole moduli of surface. Subsequently, we show, in the subsections 3.2 and 4.2, that similar notions may further be generalized for a larger number of moduli.

Interestingly, the vanishing value of the Fayet parameter comes up with an appropriate choice which arrives with the equal vacuum expectation value $\langle x_1 \rangle_0 = \langle x_2 \rangle_0$. Thus, we see that the walls of the marginal and threshold stabilities may be defined by the limiting moduli configurations which possess no Fayet term. In this case, one finds that the limit $b = 0$ provides the possible curve of the moduli stability. The situation under the present consideration may thus be described via the technology of the intrinsic Riemannian geometry, where the planar and hyperplanar stabilities are encoded in the principle minors of the corresponding covariant metric tensor. As per the comparative results of the present analysis, we find that a further application may be explored for the moduli configurations of the experimental interests, e.g. the phenomenology of string theory. In fact, there exists a list of such moduli configurations, e.g., Higgs moduli, Calabi-Yau moduli, torus moduli, instanton moduli, topological string moduli, and the moduli pertaining to D and M -particles. These examinations are however left for a separate investigation.

The rest of the paper has been organized as follows. In section 2, we provide a brief review of some of the needful concepts pertaining to the D -term potential, relevance of the Fayet models, walls of stability and their certain implications arising from the basics of the real intrinsic Riemannian geometry. In section 3, we analyze the moduli stabilization problem for the marginally stable BPS configurations and thereby explicate the nature of the stability under the transformation of the Fayet parameter. In section 4, we extend the consideration for the threshold configurations and concentrate on the examination of the underlying moduli configuration. In sections 5, we present our conclusion, and a set of perspective remarks for the future investigations.

2 D-term Potentials and Real Intrinsic Geometry

We begin by considering a brief review of the Fayet models and related concepts, which would be used in the later sections. In this section, we shall concisely provide an account for the D-term potential and stability criteria arising from the real intrinsic Riemannian geometry. The general details of the Fayet models may be found in [17, 18] and the intrinsic geometry has been explained in [85–88]. Some other possible developments and recent interesting applications of the D -term and F -term potentials may be found in Ref. [89]. What follows next is that one may consider a set of complex scalar fields $\{\varphi_i\}_{i=1}^n$ with the respective $U(1)$ -charges $\{q_i\}_{i=1}^n$. Then, the leading order D-term potential, in an appropriate normalization convention, may be expressed as,

$$V_D(\varphi_i) = \left(\sum_{i \in \Lambda_0} \varphi_i^\dagger \varphi_i + b \right)^2 \quad (2)$$

The realization of the scalar fields may be accomplished by defining $\varphi_i = x_i + ix_{i+1}$, and thus the concerned D-term potential reduces to

$$V_D(x_i) = \left(\sum_{i \in \Lambda} x_i^2 + b \right)^2 \quad (3)$$

As mentioned in the introduction, the issue under consideration may be analyzed by extremizing the D-term potential $V_D(x_i)$. The set of critical points $\mathcal{C} := \{x_i^0 \mid i = 1, 2, \dots, 2n\}$ of the potential can be easily determined by the condition $\partial V_D(x_i) = 0$. While the stability of the underlying configuration may straightforwardly be achieved by demanding the positivity of the Hessian matrix of $V_D(x_i)$. Thence, an arbitrary configuration is stabilized according to the following definition

$$\partial_i \partial_j V_D(x_i)|_{x_i=x_i^0} = g_{ij} \quad (4)$$

We may however easily notice, under the consideration of extreme moduli space geometry [17], that the ordinary Hessian matrix $\partial_i \partial_j V_D(x_i)$ defines a symmetric bilinear form, and thus supplies a real intrinsic metric tensor g_{ij} . Thus, the stability analysis of the concerned configuration may be performed, in terms of the positivity of the principle minors of the covariant metric tensor g_{ij} , see for detailed physical applications of the intrinsic geometry [36–54, 56, 60–64, 85–89].

The linear stability may simply be obtained by demanding the positivity of the principle components of the real metric tensor. Thus, the system is stable along an intended dimension n , if the respective component satisfies $g_{ij} > 0$. Furthermore, the configuration is stable on the two dimensional surfaces being defined by the coordinate chart $\{x_1, x_2\}$, if the concerned determinant of the metric tensor satisfies,

$$g_2 = g_{11}g_{22} - g_{12}^2 > 0 \quad (5)$$

Moreover, the chosen solution remains stable on the three dimensional hyper-surface, if the determinant of the metric tensor satisfies,

$$g_3 = g_{11}(g_{22}g_{33} - g_{23}^2) - g_{12}(g_{12}g_{33} - g_{13}g_{23}) + g_{13}(g_{12}g_{23} - g_{13}g_{22}) > 0 \quad (6)$$

Similarly, an arbitrary system of moduli turns out to be stable, as the $m \leq 2n$ -dimensional hyper-surface, if the concerned principle minors and the determinant of the metric tensor remain positive definite quantities. In this viewpoint, the full configuration is said to be stable against the simultaneous fluctuations of the moduli fields, if the underlying determinant of the metric tensor remains a positive definite quantity, over a range of interest of the parameters.

In the above definition, a moduli configuration is said to be completely stable, if the set

$$\mathcal{B} := \{g_{ij}, g_i, g; x_i \in M_{2n}, \forall i = 1, 2, \dots, 2n\} \quad (7)$$

remains positive definite. It is worth mentioning that a moduli is stabilized, if an arbitrary scalar $x_i \in M_{2n}$ also satisfies the same requirement, i.e. it is an element of the set \mathcal{B} .

A set of scalars $\{x_i \in M_{2n}\}$ are said to be (hyper) correlated, if the components of the underlying Riemann covariant tensor R_{ijkl} remain non-vanishing for some given indices i, j, k, l . In particular, the scalar curvature signifies an average correlation volume for the constituent moduli field configuration. The present paper concentrates on the decaying BPS configurations, which are associated with the D -term potentials. Subsequently, we shall focus our attention on the marginal and threshold walls of stabilities.

From the perspective of the D -term potential, there exists no set of critical points $\{x_1^0, x_2^0 \in \mathcal{C}\}$, such that the corresponding potential $V(x_1^0, x_2^0) = 0$. In this case, if

there are some decays allowed, then the moduli configuration thus obtained must be a non-BPS system. A similar analysis is straightforward for the multi-complex scalar configurations, as well. In particular, we may note that there does not exist a set of vacuum scalars $\{x_1^0, x_2^0, \dots, x_n^0\}$ such that the $V(x_i^0) = 0$. Consequently the daughter system must be a non-BPS configuration. Notice further that the present analysis is not limited to a single final configuration. In fact, the investigation in question may be easily carried forward for an arbitrary union of the daughter configurations, and thus for the multicentered solutions.

A possible extension of the present investigation may be accomplished, for the general non-extreme configuration, by defining the Hessian of the D-term potential as, $g_{ij} = D_i D_j V D(x_i)$, see for details [65–73]. Although, the underlying physical interpretations may not remain quite the same as those of the extreme configurations, it may however be noted that the scalar fields defining the underlying vacuum manifold may not be globally stabilized. It is hence natural to extend an understanding of the attractor flows, what we have thus studied, in the context of the simplest Fayet models, defined by the leading order D-term configurations. Nevertheless, the leading order potential function is particularly suited for the present analysis, defining an intrinsic quadratic form which assumes no supersymmetry. Thus, the present analysis considerably simplifies the underlying geometric computations and the possible investigation of the transformations concerning the Fayet parameter.

At this point, it is worth to note that our geometrical analysis may equally be applied to the other black brane solutions, which possess a definite moduli space configuration. Interestingly, the various extensions of the intrinsic geometry may further be analyzed apart from the extreme configurations. In fact, it may be noted further that the underlying investigations find a set of intriguing realizations, from the perspective of the wall crossing phenomena, for the possible values of the parameters of underlying moduli space configurations. Apart from the implications following from the leading order D -term potential, there exists a wide class of effective theories with and without the cosmological constant, extremal as well as non extremal black brane solutions, that might be further explored under the agenda of the present consideration.

The moduli space geometry has in turn given very crucial insights to the geometric understanding of a class of higher derivative corrected extremal black hole solutions. It may be instructive to pursue geometric and algebraic dispositions associated with various black hole potentials, with an inclusion of arbitrary higher derivative curvature terms, for various space-time dimensions. In particular, the underlying analysis may be investigated for the attractor stability of $AdS_2 \times S^{D-2}$ near horizon geometry, for an arbitrary extremal black hole. Importantly, the case of the Calabi-Yau black holes may investigated under the string duality transformations, containing certain monodromy invariant parameters. Thus, one may intrinsically analyze the phenomenon of the wall crossing, from the perspective of the Calabi-Yau compactification.

What follows next is that the nature of the supersymmetric moduli may be characterized by the charges carried by the configuration. Thus, depending on the sign of the underlying charges, we shall analyze the intrinsic geometric issues and thereby describe the nature of the wall crossing phenomena, for the decaying single charge and two charge moduli configurations. In the next section, we present the case of the marginal wall of stability and possible insights arising from the real intrinsic geometry.

3 Marginal Stability

In this section, we focus our attention on uniform charge distribution for the scalar fields φ_i , such that each of them has an equal unit charge. In other words, we shall consider that the charges carried by each of the scalar fields are $q_i = 1, \forall i$. To make the presentation lucid, we shall analyze the associated moduli configurations for the case of one and two $U(1)$ scalars, which respectively involve two and four real scalar fields. Sequentially, it turns out further that the general consideration may similarly be illustrated, for an arbitrary number of abelian scalar fields.

3.1 Single Complex Scalar

In order to offer a flavor of the present analysis, let us consider a single complex scalar field $\varphi_i := (x_1, x_2) \in U(1)$. Thus, with an appropriate normalization, the potential of interest may be expressed as,

$$V(x_1, x_2) := (x_1^2 + x_2^2 + b)^2 \quad (8)$$

A straightforward computation shows that the components of the metric tensor satisfy a set of quadratic polynomials,

$$\begin{aligned} g_{x_i x_i} &= 12x_i^2 + 4x_j^2 + 4b, i, j = 1, 2; i \neq j \\ g_{x_1 x_2} &= 8x_1 x_2 \end{aligned} \quad (9)$$

We thus see that the principle components of the intrinsic metric tensor are symmetric quadratic polynomials in the real scalars, whereas the off-diagonal components are symmetric quadratic monomials. From the definition of the intrinsic geometry, we see that the determinant of the metric tensor takes a well-defined positive-definite quadratic polynomial form,

$$g = 16(b^2 + 4(x_1^2 + x_2^2)b + 3(x_1^2 + x_2^2)^2) \quad (10)$$

Thence, we observe that the determinant of the metric tensor is a quadratic polynomial in the Fayet parameter defining the D- term potential of the underlying supersymmetry breaking configuration. Furthermore, we may easily compute the Christoffel connections, Riemann covariant tensors, and the associated Ricci tensors, as well. for a given intrinsic surface. Herewith, we find that the scalar curvature takes the following quotient expression,

$$R = -b \frac{(x_1^2 + x_2^2)}{(b^2 + 4(x_1^2 + x_2^2)b + 3(x_1^2 + x_2^2)^2)} \quad (11)$$

In order to understand the implication of the Fayet parameter, we shall now consider the various possible physically interesting limiting cases, and thereby focus our attention on the underlying transformations. For example, we see, in the limit $b = 0$, that the determinant of the metric tensor reduces to a positive expression

$$g = 48(x_1^2 + x_2^2)^2 \quad (12)$$

which is a positive definite function for all values of the moduli fields x_1, x_2 . Whilst, it follows that the corresponding scalar curvature vanishes identically. This implies that the underlying system becomes a non-interesting statistical system for $b = 0$.

Furthermore, under the reflection $b = -a$, we may easily notice that the determinant of the metric tensor satisfies

$$g = 16(a^2 - 4(x_1^2 + x_2^2)a + 3(x_1^2 + x_2^2)^2) \quad (13)$$

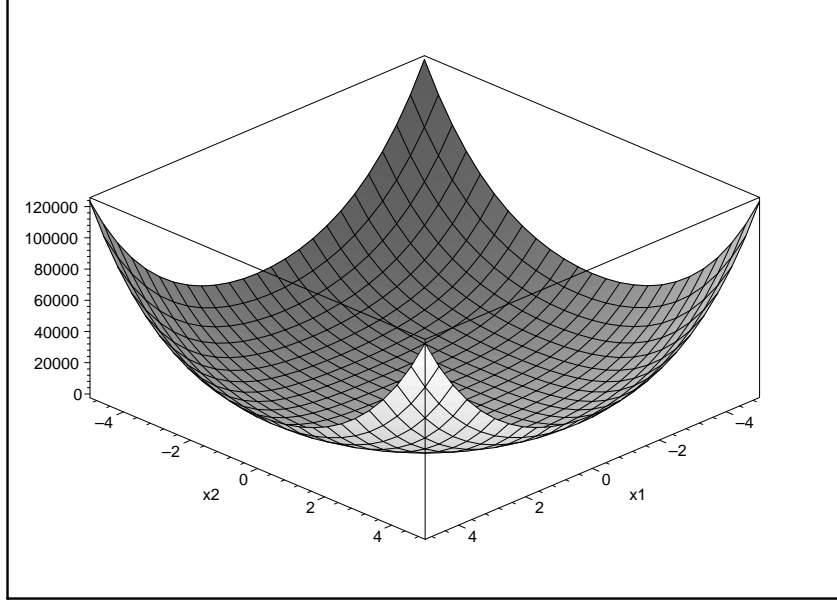


Figure 1: The determinant of the metric tensor plotted as a function of the moduli fields, x_1, x_2 , describing the fluctuations in the marginal configuration.

At such a negative value of the Fayet parameter, $b = -a$, the scalar curvature transforms to

$$R = a \frac{(x_1^2 + x_2^2)}{(a^2 - 4(x_1^2 + x_2^2)a + 3(x_1^2 + x_2^2)^2)} \quad (14)$$

This shows that the marginal configuration, as per the definition of the D -term potential, remains statistically interacting, under the reflection of the Fayet parameter.

Notice that similar conclusions are obtained under the inversion $b = 1/c$. In particular, one finds that the determinant of the metric tensor reads

$$g = \frac{16}{c^2} (cx_1^2 + cx_2^2 + 1)(3cx_1^2 + 3cx_2^2 + 1) \quad (15)$$

Correspondingly, we see that the scalar curvature reduces to

$$R = -c^3 \frac{(x_1^2 + x_2^2)}{(cx_1^2 + cx_2^2 + 1)^2 (3cx_1^2 + 3cx_2^2 + 1)^2} \quad (16)$$

Finally, it is obvious for equal values of the scalar fields, $x_1 = x$ and $x_2 = x$, that the components of the metric tensor take the following values

$$\begin{aligned} g_{x_i x_i} &= 16x^2 + 4b \\ g_{x_1 x_2} &= 8x^2 \end{aligned} \quad (17)$$

The corresponding determinant of the metric tensor simplify to the following quadratic polynomial $g = 192x^4 + 128bx^2 + 16b^2$ in b . In this limit, it is also not difficult to see that the scalar curvature reduces to the following expression

$$R = -2b \frac{x^2}{(b^2 + 8x^2b + 12x^4)^2} \quad (18)$$

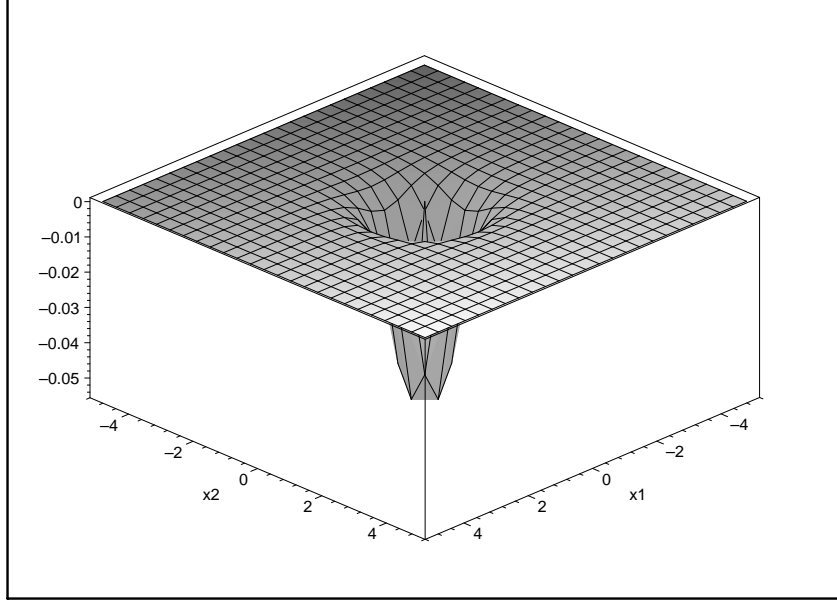


Figure 2: The scalar curvature plotted as a function of the moduli fields, x_1, x_2 , describing the fluctuations in the marginal configuration.

Interestingly, we find that the sign of the moduli interactions depends only on the sign of the Fayet parameter. Specifically, the underlying system is attractive, for a positive value of the Fayet parameter. The graphical notions of the stability and global moduli interactions are respectively depicted in the Figs(1, 2). Over a range of moduli fields $\{x_1, x_2\}$, the figures henceforth have been plotted for the choice of a unit Fayet parameter $b = 1$.

3.2 Multiple Complex Scalars

We shall now provide a general flavor of the analysis presented in the foregoing subsection. This is accomplished by taking a set of complex scalar fields $\{\varphi_i | \forall i \in \Lambda, \varphi_i \in U(1)\}$, where the index Λ is taken to be the set of a finite cardinality. First of all, it is natural to concentrate on the case of the two complex scalar fields $\{\varphi_1, \varphi_2\}$, thus applying the previously defined analysis for the four real scalars (x_1, x_2, x_3, x_4) . Nevertheless, it may be further envisaged that it is possible to analyze the nature of the marginal stability for the general moduli configurations with an arbitrary Λ , as well. Let us focus our attention on the marginal stability of the leading order configurations. In particular, let us consider the D-term potential as a function of the four real scalar fields, viz., $x_a = (x_1, x_2, x_3, x_4)$. Thus, it is immediate to see that the underlying potential may be expressed as,

$$V(x_1, x_2, x_3, x_4) := (x_1^2 + x_2^2 + x_3^2 + x_4^2 + b)^2 \quad (19)$$

A straightforward calculation implies that the covariant components of the metric tensor are,

$$\begin{aligned} g_{x_i x_i} &= 12x_i^2 + 4 \sum_{j \neq i} x_j^2 + 4b, \quad i = 1, 2, 3, 4 \\ g_{x_i x_j} &= 8x_i x_j, \quad \forall i \neq j, \quad i, j = 1, 2, 3, 4 \end{aligned} \quad (20)$$

In this case, we observe further that the principle components of the intrinsic metric tensor are quadratic polynomials in the real scalars, while the diagonal components are symmetric monomials. Apart from the number of polynomials, this property remains exactly the same, as mentioned in the foregoing subsection. In order to analyze the planar and hyper planar stability conditions, we now need to consider the principle minors of the metric tensor. Thence, we find that the planar principle minor, defined as $g_2 := g_{11}g_{22} - g_{22}^2$, leads to the following quadratic polynomial,

$$g_2 := g_{22}b^2 + g_{21}b + g_{20} \quad (21)$$

where the coefficients of the above equation are given by the following expressions

$$\begin{aligned} g_{22} &:= 16 \\ g_{21} &:= (32x_3^2 + 32x_4^2 + 64x_1^2 + 64x_2^2) \\ g_{20} &:= 64x_2^2x_3^2 + 64x_2^2x_4^2 + 96x_2^2x_1^2 + 48x_2^4 + 16x_3^4 \\ &\quad + 16x_4^4 + 48x_1^4 + 64x_1^2x_4^2 + 64x_1^2x_3^2 + 32x_3^2x_4^2 \end{aligned} \quad (22)$$

Whilst, the hyper-planar stability may be examined by the third principle minor, defined as $g_3 = g_{11}(g_{22}g_{33} - g_{23}^2) - g_{12}(g_{12}g_{33} - g_{13}g_{23}) + g_{13}(g_{12}g_{23} - g_{13}g_{22})$. It is not difficult to show that g_3 simplifies to a cubic polynomial

$$g_3 := g_{33}b^3 + g_{32}b^2 + g_{31}b + g_{30} \quad (23)$$

where the coefficients of the above equation take the following expressions

$$\begin{aligned} g_{33} &:= 64 \\ g_{32} &:= 192x_4^2 + 320x_1^2 + 320x_2^2 + 320x_3^2 \\ g_{31} &:= 448x_3^4 + 384x_1^2x_4^2 + 192x_4^4 + 1152x_1^2x_3^2 + 640x_2^2x_4^2 \\ &\quad + 896x_2^2x_1^2 + 448x_2^4 + 896x_2^2x_3^2 + 448x_1^4 + 640x_3^2x_4^2 \\ g_{30} &:= 896x_1^2x_2^2x_3^2 + 576x_1^2x_2^4 + 832x_1^2x_3^4 + 64x_1^2x_4^4 + 832x_1^4x_3^2 \\ &\quad + 192x_1^4x_4^2 + 576x_1^4x_2^2 + 576x_2^4x_3^2 + 448x_2^4x_4^2 + 448x_3^4x_2^2 \\ &\quad + 576x_2^2x_3^4 + 320x_2^2x_4^4 + 192x_3^6 + 64x_4^6 + 1024x_2^2x_1^2x_4x_3 \\ &\quad + 192x_1^6 + 192x_2^6 + 320x_3^2x_4^4 + 128x_1^2x_2^2x_4^2 + 896x_2^2x_3^2x_4^2 \\ &\quad + 896x_1^2x_3^2x_4^2 \end{aligned} \quad (24)$$

For arbitrary values of the scalar fields and the Fayet parameter, it is not difficult to see that the determinant of the metric tensor is given by,

$$g := g_{44}b^4 + g_{43}b^3 + g_{42}b^2 + g_{41}b + g_{40} \quad (25)$$

where the coefficients of the above equation can be expressed as follows

$$\begin{aligned} g_{44} &:= 256 \\ g_{43} &:= 1536x_3^2 + 1536x_2^2 + 1536x_4^2 + 1536x_1^2 \\ g_{42} &:= 3072x_3^4 + 6144x_1^2x_3^2 + 6144x_1^2x_4^2 + 3072x_2^4 + 6144x_2^2x_3^2 \\ &\quad + 3072x_1^4 + 6144x_2^2x_1^2 + 6144x_3^2x_4^2 + 3072x_4^4 + 6144x_2^2x_4^2 \\ g_{41} &:= 7680x_4^2x_3^2 + 2560x_3^6 + 7680x_1^4x_3^2 + 7680x_2^4x_3^2 + 2560x_2^6 \\ &\quad + 7680x_1^2x_4^4 + 7680x_3^4x_4^2 + 7680x_1^4x_4^2 + 7680x_2^2x_3^4 + 15360x_1^2x_3^2x_4^2 \\ &\quad + 15360x_1^2x_2^2x_3^2 + 7680x_1^2x_3^4 + 7680x_1^2x_4^4 + 2560x_4^6 + 7680x_3^2x_4^4 \\ &\quad + 2560x_1^6 + 7680x_2^2x_4^4 + 15360x_1^2x_2^2x_4^2 + 7680x_1^4x_2^2 + 15360x_2^2x_3^2x_4^2 \\ g_{40} &:= 3072x_2^6x_3^2 + 4608x_2^4x_3^4 + 4608x_2^4x_4^4 + 3072x_2^6x_4^2 + 3072x_2^2x_3^6 \end{aligned}$$

$$\begin{aligned}
& +3072x_2^2x_4^6 + 4608x_1^4x_2^4 + 3072x_1^2x_2^6 + 3072x_1^6x_2^2 + 3072x_4^6x_3^2 \\
& +4608x_3^4x_4^4 + 3072x_3^6x_4^2 + 4608x_1^4x_3^4 + 3072x_1^2x_3^6 + 3072x_1^6x_3^2 \\
& +4608x_1^4x_4^4 + 3072x_1^6x_3^2 + 3072x_1^2x_4^6 + 9216x_1^2x_3^4x_4^2 + 9216x_1^4x_2^2x_3^2 \\
& +9216x_1^2x_2^4x_3^2 + 9216x_1^2x_2^2x_3^4 + 9216x_1^2x_4^4x_2^2 + 9216x_1^2x_4^2x_3^2 \\
& +9216x_1^4x_3^2x_4^2 + 9216x_1^4x_4^2x_2^2 + 9216x_2^4x_3^2x_4^2 + 9216x_2^2x_4^4x_1^2 \\
& +9216x_2^2x_3^4x_4^2 + 9216x_2^2x_4^4x_3^2 + 768x_1^8 + 768x_2^8 + 768x_4^8 \\
& +768x_3^8 + 18432x_2^2x_1^2x_4^2x_3^2
\end{aligned} \tag{26}$$

An analogous analysis may as well be performed easily for the concerned global intrinsic geometric invariant quantities. In particular, it is not difficult to see that the underlying Ricci scalar of the fluctuations is,

$$R = -\frac{9}{2} \frac{\sum_{i=1}^4 x_i^2}{(b + 3 \sum_{j=1}^4 x_j^2)} \frac{1}{(b^2 + r_{m1}b + r_{m0})} \tag{27}$$

where the polynomial functions r_{m1} and r_{m0} are defined as follows

$$\begin{aligned}
r_{m1} &:= 4 \left(\sum_{i=1}^4 x_i^2 \right) \\
r_{m0} &:= 3 \left(\sum_{i=1}^4 x_i^2 \right)^2
\end{aligned} \tag{28}$$

For a pairwise equal value of the moduli fields, viz., $x_1 = x_2 = x$, $x_3 = x_4 = y$ and for a positive value of the Fayet parameter, e.g., $b = 1$, the principle minors are given by

$$g_2 = 256x^2y^2 + 192x^4 + 16 + 64y^2 + 64y^4 + 128x^2 \tag{29}$$

$$\begin{aligned}
g_3 &= 64 + 640x^2 + 512y^2 + 3072x^2y^2 + 1792x^4 \\
&+ 1280y^4 + 4096x^4y^2 + 3584x^2y^4 + 1024y^6 + 1536x^6
\end{aligned} \tag{30}$$

$$\begin{aligned}
g &= 256 + 3072x^2 + 3072y^2 + 24576x^2y^2 + 12288x^4 + 12288y^4 \\
&+ 49152x^2y^6 + 73728x^4y^4 + 49152x^6y^2 + 61440x^4y^2 \\
&+ 61440x^2y^4 + 12288x^8 + 12288y^8 + 20480y^6 + 20480x^6
\end{aligned} \tag{31}$$

It follows further that the associated scalar curvature is given by the following formula

$$R = -9 \frac{(x^2 + y^2)}{(1 + 6x^2 + 6y^2)(1 + 8y^2 + 8x^2 + 12x^4 + 12y^4 + 24x^2y^2)} \tag{32}$$

The qualitative behavior of the principle minors g_2, g_3 , determinant of the metric tensor g and the scalar curvature R has been respectively shown in the following 3 dimensional figures: Figs.(3, 4, 5, 6).

For equal values of the scalar fields, viz., $x = z$, $y = z$ and $b = 1$, the limiting determinant of the metric tensor possesses the following qualitative behavior. Specifically, the two dimensional nature of the determinant of the metric tensor g and the scalar curvature R has been respectively shown in the following plots: Figs.(7, 8).

For equal values of the scalar fields, $x_i = x$, we find, for any general Fayet parameter b , that the planar stability may be determined from the polynomial

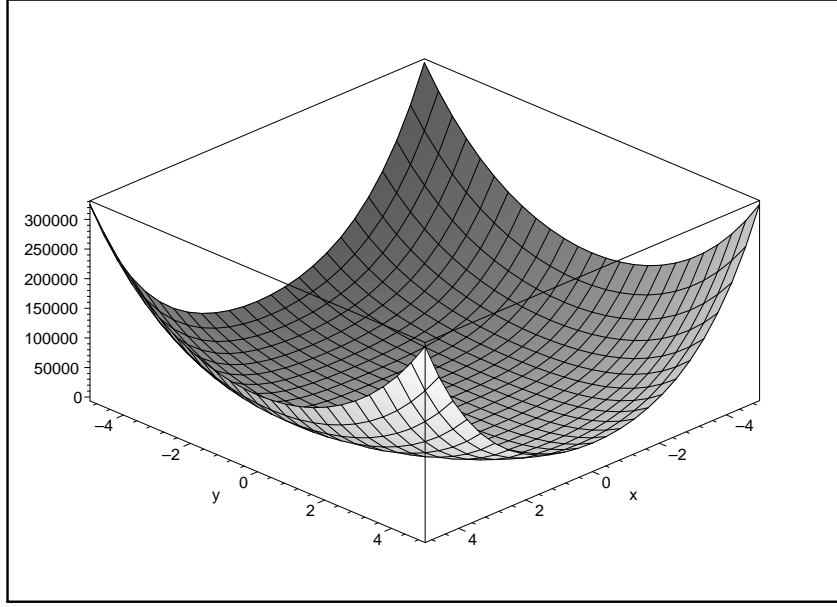


Figure 3: The surface minor plotted as a function of the moduli fields, x, y , describing the fluctuations in the marginal configuration.

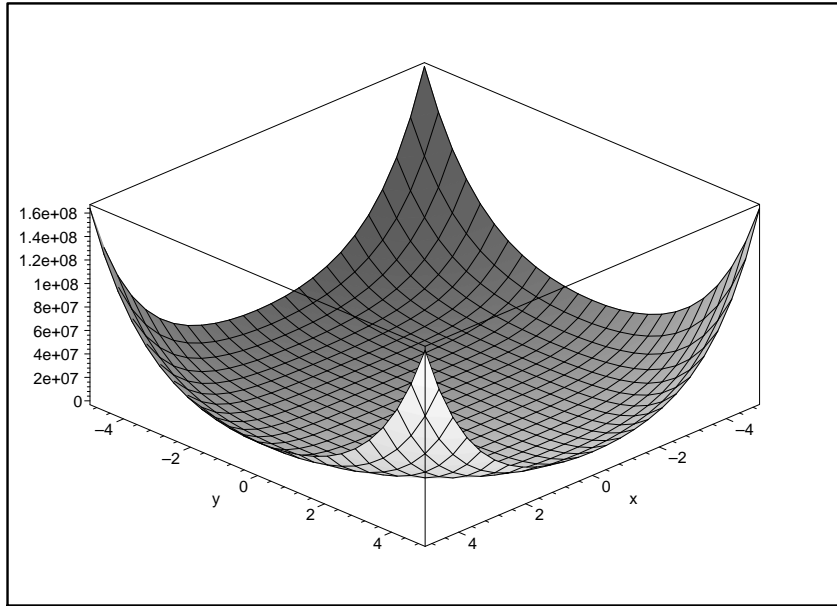


Figure 4: The hypersurface minor plotted as a function of the moduli fields, x, y , describing the fluctuations in the marginal configuration.

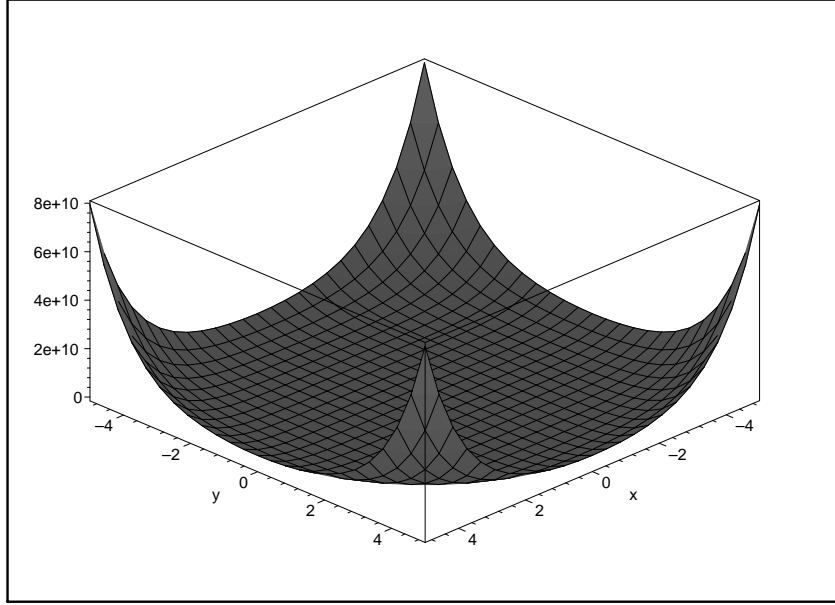


Figure 5: The determinant of the metric tensor plotted as a function of the moduli fields, x, y , describing the fluctuations in the marginal configuration.

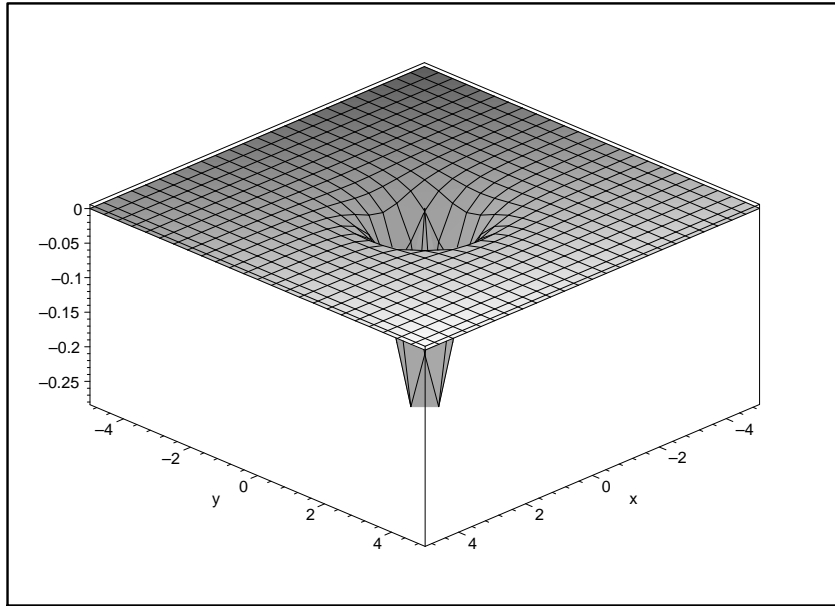


Figure 6: The scalar curvature plotted as a function of the moduli fields, x, y , describing the fluctuations in the marginal configuration.

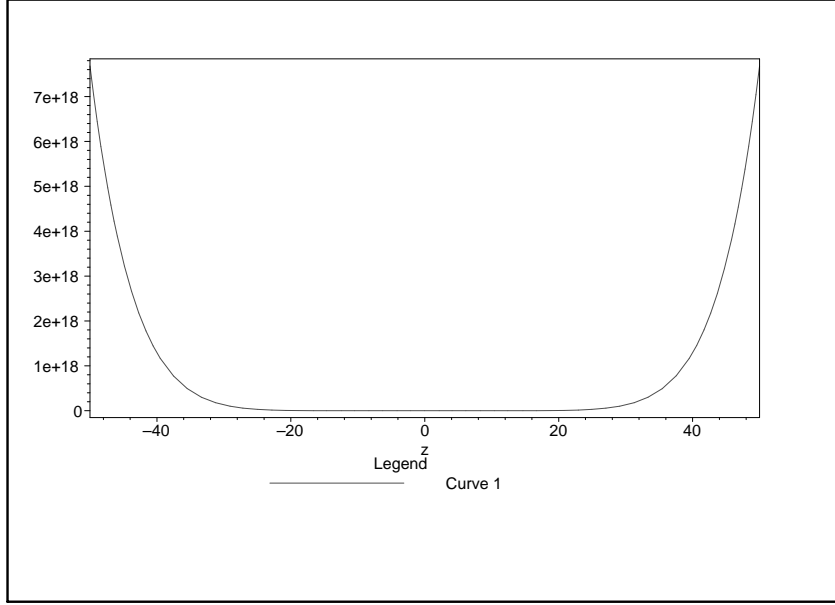


Figure 7: The determinant of the metric tensor plotted as a function of the moduli fields, z , describing the fluctuations in the marginal configuration.

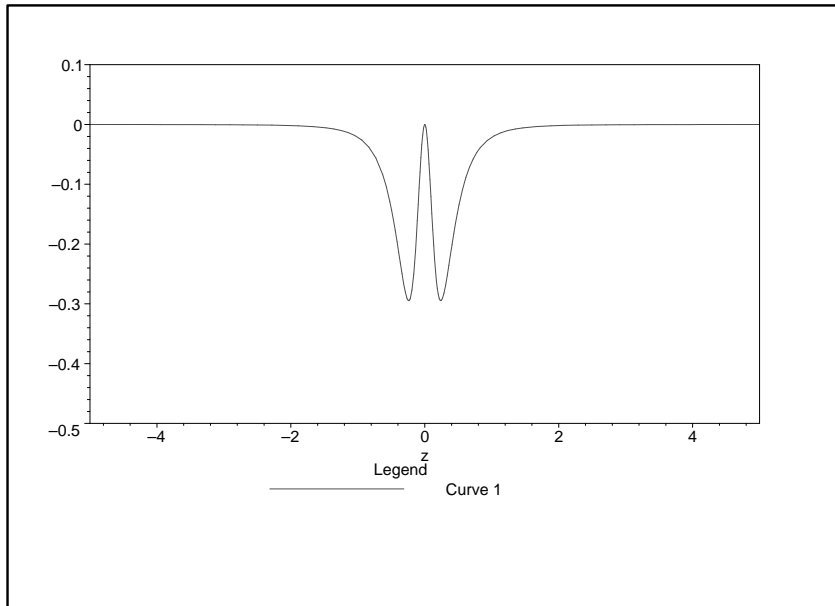


Figure 8: The scalar curvature plotted as a function of the moduli fields, z , describing the fluctuations in the marginal configuration.

expressions of the corresponding principle minors, viz., $\{g_2, g_3\}$ and the determinant of the metric tensor g , as the highest principle minor. In this case, it turns out that the respective limiting values of the principle minors and determinant of the metric tensor are given by

$$g_2 := 16b^2 + 192x^2b + 512x^4 \quad (33)$$

$$g_3 := 64b^3 + 1152x^2b^2 + 6144x^4b + 10240x^6 \quad (34)$$

$$g := 256b^4 + 6144x^2b^3 + 49152x^4b^2 + 163840x^6b + 196608x^8 \quad (35)$$

The associated scalar curvature is given by the

$$R = -18 \frac{x^2}{(12x^2 + b)(b^2 + 16x^2b + 48x^4)} \quad (36)$$

It may thus be noted that the restricted two dimensional moduli configuration has the same degree of the polynomial in the Fayet parameter, as that of the moduli configuration with a single complex scalar field. Nevertheless, the stability of the two moduli configuration requires the positivity of the hyper-planar minor, viz., g_3 and that of the other higher principle minors, e.g. determinant of the metric tensor g .

Such stability constraints may further be understood from the fact that the three dimensional hyper-surface satisfies a third degree polynomial equation in the Fayet parameter b .

For the case of the limiting Fayet parameter $b = 0$, we herewith observe that the local moduli pair correlations reduce to the following expressions

$$\begin{aligned} g_{x_i x_i} &= 12x_i^2 + 4 \sum_{j \neq i} x_j^2 \\ g_{x_i x_j} &= 8x_i x_j, i \neq j, i, j = 1, 2, 3, 4 \end{aligned} \quad (37)$$

In this case, we find that the surface minor take the following pure quartic form

$$\begin{aligned} g_2 &= 64x_2^2x_3^2 + 64x_2^2x_4^2 + 96x_2^2x_1^2 + 48x_2^4 + 16x_3^4 \\ &\quad + 16x_4^4 + 48x_1^4 + 64x_1^2x_4^2 + 64x_1^2x_3^2 + 32x_3^2x_4^2 \end{aligned} \quad (38)$$

For the given moduli fields $\{x_1, x_2, x_3, x_4\}$, the hypersurface minor is given by the following four variable homogeneous degree 6 polynomial

$$\begin{aligned} g_3 &= 896x_1^2x_2^2x_3^2 + 576x_1^2x_2^4 + 832x_1^2x_3^4 + 64x_1^2x_4^4 + 832x_1^4x_2^2 \\ &\quad + 192x_1^4x_3^2 + 576x_1^4x_4^2 + 576x_2^4x_3^2 + 448x_2^4x_4^2 + 448x_3^4x_4^2 \\ &\quad + 576x_2^2x_3^4 + 320x_2^2x_4^4 + 192x_3^6 + 64x_4^6 + 1024x_2^2x_1^2x_4x_3 \\ &\quad + 192x_1^6 + 192x_2^6 + 320x_3^2x_4^4 + 128x_1^2x_2^2x_4^2 + 896x_2^2x_3^2x_4^2 \\ &\quad + 896x_1^2x_3^2x_4^2 \end{aligned} \quad (39)$$

For the vanishing Fayet parameter $b = 0$, we may now observe that the determinant of the metric tensor reduces to the following homogeneous degree 8 polynomial

$$\begin{aligned} g &= 9216x_2^4x_4^2x_1^2 + 9216x_1^2x_4^4x_2^2 + 18432x_2^2x_1^2x_4^2x_3^2 + 9216x_1^2x_2^2x_3^4 \\ &\quad + 3072x_2^6x_3^2 + 4608x_2^4x_3^4 + 4608x_2^4x_4^4 + 3072x_2^6x_4^2 + 3072x_2^2x_3^6 \\ &\quad + 3072x_2^2x_4^6 + 4608x_1^4x_2^2 + 3072x_1^2x_2^6 + 3072x_1^6x_2^2 + 3072x_1^4x_3^2 \\ &\quad + 4608x_1^4x_4^2 + 3072x_1^6x_3^2 + 4608x_1^4x_3^4 + 3072x_1^2x_3^6 + 3072x_1^2x_4^6 \\ &\quad + 9216x_2^2x_4^4x_3^2 + 9216x_1^4x_4^2x_2^2 + 9216x_1^2x_4^4x_2^2 + 9216x_1^2x_3^4x_2^2 \\ &\quad + 9216x_1^2x_4^4x_3^2 + 9216x_1^4x_3^2x_4^2 + 9216x_2^4x_3^2x_4^2 + 9216x_2^2x_3^4x_4^2 \\ &\quad + 4608x_1^4x_4^4 + 3072x_1^6x_3^2 + 3072x_1^6x_4^2 + 768x_1^8 + 768x_2^8 \\ &\quad + 768x_3^8 + 768x_4^8 + 9216x_1^4x_2^2x_3^2 \end{aligned} \quad (40)$$

Similarly, the corresponding scalar curvature is given by the following expression

$$R = -\frac{1}{2} \frac{1}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2}. \quad (41)$$

Herewith, we find that the scalar curvature may be described as an inverse square of the following pure quadratic form $\sum_{i=1}^4 x_i^2$. This shows that the limiting global moduli interactions, following from the intrinsic geometric scalar curvature, depend only on the sum of the squares of the constituent moduli fields.

For the vanishing value of the Fayet parameter $b = 0$, it turns out that the planar and hyperplanar stabilities are well ensured, in the limit of equal values of the moduli fields, viz., $x_i = x$. In particular, we notice, for the case of $b = 0$, that the principle minors $\{g_2, g_3\}$ correspond to the following positive monomial expressions, $g_2 := 512x^4$ and $g_3 := 10240x^6$. Furthermore, we find, for equal values of the real scalars $x_i = x$ and vanishing Fayet parameter $b = 0$, that the determinant of the underlying metric tensor reduces to the following positive monomial $g = 196608x^8$. Thus, we find that the limiting equal moduli marginal configuration remains stable under all possible fluctuations of the moduli x .

On the other hand, we see, under the reflection of the Fayet parameter $b = -a$, that the components of the metric tensor reduce to the following expressions

$$\begin{aligned} g_{x_i x_i} &= 12x_i^2 + 4 \sum_{j \neq i} x_j^2 - 4a \\ g_{x_i x_j} &= 8x_i x_j, \quad i \neq j, \quad i, j = 1, 2, 3, 4 \end{aligned} \quad (42)$$

Furthermore, it turns out, under the transformation $b = -a$, that the corresponding planar and hyperplanar stability constraints are rearranged as per the following general polynomials,

$$g_2 := 16a^2 - g_{21}a + g_{20} \quad (43)$$

$$g_3 := -64a^3 + g_{32}a^2 - g_{31}a + g_{30} \quad (44)$$

$$g := g_{44}a^4 - g_{43}a^3 + g_{42}a^2 - g_{41}a + g_{40} \quad (45)$$

In this case, we notice that the scalar curvature transforms into the following expression

$$R := \frac{9}{2} \frac{(\sum_{i=1}^4 x_i^2)}{(a - 3 \sum_{i=1}^4 x_i^2)} \frac{1}{(a^2 - r_{m1}a + r_{m0})} \quad (46)$$

Under the inverse of the Fayet parameter $b := 1/c$, we observe that the local pair correlations scale as

$$\begin{aligned} g_{x_i x_i} &= \frac{4}{c} ((3x_i^2 + \sum_{i \neq j} x_j^2)c + 1) \\ g_{x_i x_j} &= 8x_i x_j, \quad \forall i \neq j, \quad i, j = 1, 2, 3, 4 \end{aligned} \quad (47)$$

Thus, the inverse transformation $b = 1/c$ indicates that the corresponding values of the principle minors transform according to the root-set of the underlying polynomial equations. In this case, we see that the principle minors get inverted in the Fayet parameter, viz., the associated coefficients of the principle minors get symmetrically exchanged in those of the concerned polynomial expressions. In particular, we find that the principle minors transform according to

$$g_2 := \frac{1}{c^2} (g_{22} + g_{21}c + g_{20}c^2) \quad (48)$$

$$g_3 := \frac{1}{c^3}(g_{33} + g_{32}c + g_{31}c^2 + g_{30}c^3) \quad (49)$$

$$g = \frac{1}{c^4}(g_{44} + g_{43}c + g_{42}c^2 + g_{41}c^3 + g_{40}c^4) \quad (50)$$

Under the inversion of the Fayet parameter, we may notice further that there exists a similar scaling property for the associated intrinsic scalar curvature. We observe that the scalar curvature transforms into the following expression

$$R = -\frac{9}{2}c^3 \frac{\sum_{i=1}^4 x_i^2}{(1 + 3c \sum_{i=1}^4 x_i^2)} \frac{1}{(r_{m0}c^2 + r_{m1}c + 1)}, \quad (51)$$

where the coefficients r_{m0} and r_{m1} remain unchanged, under the dilatation $b = 1/c$.

4 Threshold Stability

In this section, we shall analyze threshold stability conditions and thereby provide an account of the BPS configuration, which can decay to an intriguing set of other moduli configurations. From the perspective of the supergravity constraints, it is known that the charges of the theory must be chosen, such that the effective potential vanishes at least for some specific value of the Fayet parameter. Thus, one needs to choose at least two abelian scalar fields, in order to preserve some supersymmetry. Before proceeding to the general consideration, we shall examine an associated interesting threshold configuration of two $U(1)$ scalars, which involves only the two real scalar fields. In the next subsection, we shall offer stability properties for such a configuration of the moduli fields.

4.1 Simple Complex Scalars

Let us consider two complex scalar fields $\{\varphi_1, \varphi_2\}$ and focus our attention on the specific subsector of the moduli configuration which involves only the two real scalars. We may easily see that the general choice of the present interest may be defined as an element $\{\varphi_1, \varphi_2\} \in \mathcal{M}$, such that the D -term potential vanishes at the vacuum. In fact, it is not difficult to see that the possible values of the fields are an element of \mathcal{M} which may, in extreme, be defined as the following set,

$$\mathcal{M} := \{(x_1, 0), (x_2, 0)\}, \{(x_1, 0), (0, x_2)\}, \{(0, x_1), (x_2, 0)\}, \{(0, x_1), (0, x_2)\} \quad (52)$$

In order to ensure supersymmetry, we may be required to choose two alternating abelian charges, which in an appropriate normalization correspond to $q_1 = 1$ and $q_2 = -1$. Whence, the threshold stability of the D -term moduli configurations may be specified by the following Fayet like potential,

$$V(x_1, x_2) := (x_1^2 - x_2^2 + b)^2 \quad (53)$$

The components of the covariant metric tensor may thus be expressed as,

$$\begin{aligned} g_{x_1 x_1} &= 12x_1^2 - 4x_2^2 + 4b \\ g_{x_1 x_2} &= -8x_2 x_1 \\ g_{x_2 x_2} &= 12x_2^2 - 4x_1^2 - 4b \end{aligned} \quad (54)$$

It is straightforward to show that the determinant of the metric tensor is

$$g = -16(b^2 - 4(x_1^2 - x_2^2)b - 3(x_1^2 - x_2^2)^2) \quad (55)$$

As in the case of the marginally stable configurations, we find in the present case that the Ricci scalar takes the following form,

$$R = -b \frac{(x_1^2 - x_2^2)}{(b^2 - 4(x_1^2 - x_2^2)b - 3(x_1^2 - x_2^2)^2)} \quad (56)$$

Notice that the scalar curvature involves only even powers of the moduli fields. Thus, the global moduli interactions remain intact under the reflection of the moduli fields $\{x_1, x_2\}$. For the case of the vanishing Fayet parameter $b = 0$, we have the following local pair correlations

$$\begin{aligned} g_{x_1 x_1} &= 12x_1^2 - 4x_2^2 \\ g_{x_1 x_2} &= -8x_2 x_1 \\ g_{x_2 x_2} &= 12x_2^2 - 4x_1^2 \end{aligned} \quad (57)$$

From the above equations, we observe that the components of the metric tensor are symmetric under the exchange of the moduli fields, viz., we have $g_{x_1 x_1} = g_{x_2 x_2}$, under the replacement of the moduli field x_1 by x_2 , and vice-versa. Furthermore, the determinant of the metric tensor reduces to a pure quartic form: $g = 96x_1^2 x_2^2 - 48x_1^4 - 48x_2^4$, while the scalar curvature vanishes identically. In this case, we find a series of striking notions of the wall crossing pertaining to the D -term potentials. Specifically, the value of the Fayet parameter b ascribes the nature of the walls of stability as the polynomial invariance of the intrinsic geometric configuration.

On the other hand, we see, for equal values of the real scalars $x_1 = x_2 = x$, that the components of the metric tensor reduce to the following asymmetric local correlations

$$\begin{aligned} g_{x_1 x_1} &= 8x^2 + 4b \\ g_{x_1 x_2} &= -8x^2 \\ g_{x_2 x_2} &= 8x^2 - 4b \end{aligned} \quad (58)$$

It turns out that the above asymmetry leads to an unstable configuration of the moduli fields. The reason follows from the fact that the underlying determinant of the metric tensor reduces to a negative value $g = -16b^2$. In this case, we notice further that the scalar curvature vanishes identically. It may thus be envisaged, for equal values of the moduli fields $x_1 = x_2 = x$, that the D -term moduli configurations become uniformly unstable and statistically non-interacting, during the limiting threshold transitions in the moduli fields $\{x_1, x_2\}$.

Under the reflection of the Fayet parameter, viz., $b = -a$, we find that the determinant of the metric tensor reduces to

$$g = -16(a^2 - (4x_1^2 - 4x_2^2)a - 6x_2^2 x_1^2 + 3x_1^4 + 3x_2^4) \quad (59)$$

while the scalar curvature reads

$$R = a \frac{(x_1^2 - x_2^2)}{(a^2 - (4x_1^2 - 4x_2^2)a - 6x_2^2 x_1^2 + 3x_1^4 + 3x_2^4)^2} \quad (60)$$

under the reflection transformation $b = -a$.

Under the inversion of the Fayet parameter, viz., $b = 1/c$, it is easy to observe that the determinant of the metric tensor may easily be expressed as,

$$g = -\frac{16}{c^2} (3(x_1^2 - x_2^2)^2 c^2 + 4(x_1^2 - x_2^2)c + 1) \quad (61)$$

The respective value of the scalar curvature is given by

$$R = -c^3 \frac{(x_1^2 - x_2^2)}{((3(x_1^2 - x_2^2)^2 c^2 + 4(x_1^2 - x_2^2)c + 1))^2} \quad (62)$$

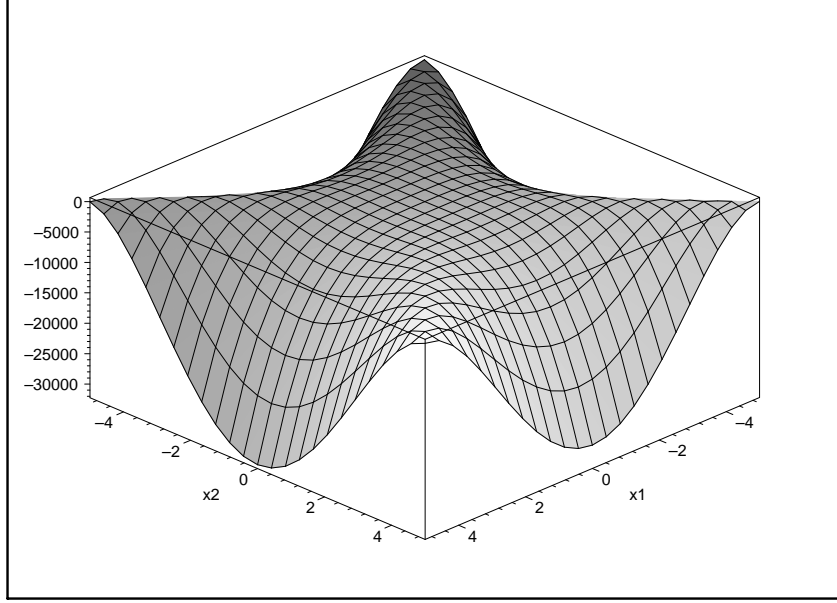


Figure 9: The determinant of the metric tensor plotted as a function of the moduli fields, x_1, x_2 , describing the fluctuations in the marginal configuration.

We thus notice that the dilatation of b keeps the singularity structure intact, except for the case when the Fayet parameter vanishes, viz., we have avoided the inversion of the limit $b = 0$. The graphical viewpoint of the moduli interaction is depicted in Figs.(9, 10). In order to make a qualitative comparison with the outcomes of the previous section, the figures of the present interests are also plotted for the unit Fayet parameter $b = 1$.

4.2 General Complex Scalars

In this subsection, we shall provide a detailed analysis for the most general two complex moduli configuration involving two $U(1)$ fields and, in general, four real scalar fields. Firstly, we shall concentrate on the two complex scalar fields $\{\varphi_1, \varphi_2 \in U(1)\}$ and supply an intriguing stability analysis, in terms of respective real scalars (x_1, x_2, x_3, x_4) . Furthermore, it turns out that one may as well analyze the nature of D-terms threshold stability configurations for the set of general moduli $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$. Nevertheless, the threshold stability of general two charge BPS configurations may be investigated by prescribing four real scalar fields, $x_a = (x_1, x_2, x_3, x_4)$ ascribing the D-term potential,

$$V(x_1, x_2, x_3, x_4) := (x_1^2 + x_2^2 - x_3^2 - x_4^2 + b)^2 \quad (63)$$

For the given moduli fields (x_1, x_2, x_3, x_4) , it is however not difficult to show that the components of the covariant metric tensor may be expressed as,

$$\begin{aligned} g_{x_i x_i} &= 12x_i^2 + 4x_j^2 - 4 \sum_{k \neq i, j} x_k^2 + 4b, i, j = 1, 2 \\ g_{x_i x_i} &= 12x_i^2 + 4x_j^2 - 4 \sum_{k \neq i, j} x_k^2 - 4b, i, j = 3, 4 \\ g_{x_i x_j} &= 8x_i x_j, (i, j) \in \{(1, 2), (3, 4)\} \\ g_{x_i x_j} &= -8x_i x_j, (i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\} \end{aligned} \quad (64)$$

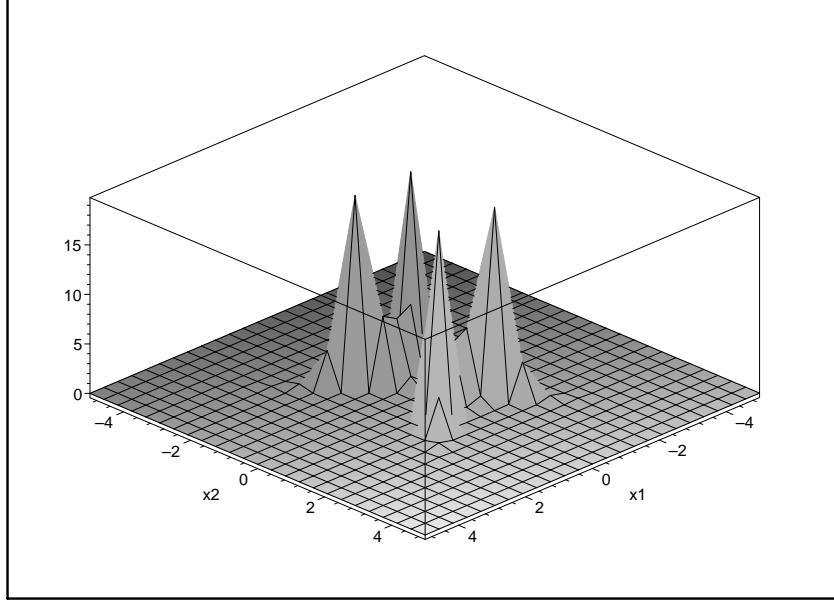


Figure 10: The scalar curvature plotted as a function of the moduli fields, x_1, x_2 , describing the fluctuations in the marginal configuration.

The planar and hyper planar stability may now be analyzed, as well. In order to do so, we would need to consider the principle minors of the underlying intrinsic metric tensor. In fact, one may find that the planar principle minor, defined as $g_2 := g_{11}g_{22} - g_{22}^2$, leads to the following quadratic polynomial

$$g_2 := g_{22}b^2 + g_{21}b + g_{20} \quad (65)$$

where the coefficients are given by the combinations

$$\begin{aligned} g_{22} &:= 16 \\ g_{21} &:= (64x_2^2 - 32x_3^2 + 64x_1^2 - 32x_4^2) \\ g_{20} &:= 64x_1^2x_3^2 + 96x_2^2x_1^2 + 16x_4^4 - 64x_2^2x_4^2 + 48x_2^4 \\ &\quad + 16x_3^4 + 32x_3^2x_4^2 + 48x_1^4 - 64x_2^2x_3^2 - 64x_1^2x_4^2 \end{aligned} \quad (66)$$

Furthermore, a straightforward computation thence shows that the hyper-planar principle minor, defined as $g_3 = g_{11}(g_{22}g_{33} - g_{22}^2)g_{12}(g_{12}g_{33} - g_{13}g_{23}) + g_{13}(g_{12}g_{23} - g_{13}g_{22})$, reduces to the following cubic polynomial

$$g_3 := g_{33}b^3 + g_{32}b^2 + g_{31}b + g_{30} \quad (67)$$

where the coefficients of the cubic equation are given by the expressions

$$\begin{aligned} g_{33} &:= -64 \\ g_{32} &:= 320x_3^2 + 192x_4^2 - 320x_1^2 - 320x_2^2 \\ g_{31} &:= 896x_2^2x_3^2 - 448x_3^4 + 640x_2^2x_4^2 - 448x_2^4 \\ &\quad - 448x_1^4 + 896x_1^2x_3^2 + 640x_1^2x_4^2 - 192x_4^4 \\ &\quad - 640x_3^2x_4^2 - 896x_2^2x_1^2 \\ g_{30} &:= 896x_1^2x_2^2x_4^2 - 576x_1^4x_2^2 + 448x_1^4x_4^2 - 576x_1^2x_2^4 \\ &\quad + 1152x_1^2x_2^2x_3^2 - 896x_1^2x_3^2x_4^2 - 192x_2^6 + 64x_4^6 \end{aligned}$$

$$\begin{aligned}
& +192x_3^6 - 192x_1^6 - 320x_1^2x_4^4 - 320x_2^2x_4^4 + 576x_1^4x_3^2 \\
& +448x_2^4x_4^2 + 320x_3^2x_4^4 - 576x_1^2x_3^4 + 576x_2^4x_3^2 \\
& -576x_2^2x_3^4 - 896x_2^2x_3^2x_4^2 + 448x_3^4x_4^2
\end{aligned} \tag{68}$$

It is not difficult to observe that the determinant of the metric tensor is given by the following quartic polynomial

$$g := g_{44}b^4 + g_{43}b^3 + g_{42}b^2 + g_{41}b + g_{40} \tag{69}$$

where the coefficients appearing in the above equation read

$$\begin{aligned}
g_{44} &:= 256 \\
g_{43} &:= 1536x_2^2 - 1536x_3^2 - 1536x_4^2 + 1536x_1^2 \\
g_{42} &:= 3072x_4^4 + 3072x_1^4 - 6144x_1^2x_4^2 + 3072x_2^4 \\
&+ 3072x_3^4 - 6144x_2^2x_4^2 - 6144x_1^2x_3^2 + 6144x_3^2x_4^2 \\
&- 6144x_2^2x_3^2 + 6144x_2^2x_1^2 \\
g_{41} &:= -15360x_1^2x_2^2x_3^2 - 2560x_3^6 + 7680x_2^2x_3^4 - 15360x_1^2x_2^2x_4^2 \\
&- 2560x_4^6 + 2560x_2^6 - 7680x_2^4x_4^2 - 7680x_2^4x_3^2 - 7680x_1^4x_4^2 \\
&+ 7680x_1^2x_2^4 - 7680x_3^4x_4^2 + 7680x_1^2x_4^4 + 7680x_1^2x_3^4 \\
&+ 7680x_1^4x_2^2 + 7680x_2^2x_4^4 + 2560x_1^6 - 7680x_3^2x_4^4 \\
&+ 15360x_2^2x_3^2x_4^2 + 15360x_1^2x_3^2x_4^2 - 7680x_1^4x_3^2 \\
g_{40} &:= 4608x_3^4x_4^4 + 9216x_2^4x_4^2x_3^2 - 9216x_2^2x_3^2x_4^2 \\
&- 9216x_2^2x_4^4x_3^2 + 768x_3^8 + 3072x_3^6x_4^2 - 3072x_3^6x_1^2 \\
&- 3072x_3^6x_2^2 - 3072x_1^6x_4^2 + 3072x_1^6x_2^2 - 3072x_1^6x_3^2 \\
&- 3072x_2^6x_4^2 + 3072x_2^6x_1^2 - 3072x_2^6x_3^2 - 3072x_4^6x_1^2 \\
&- 3072x_4^6x_2^2 + 3072x_4^6x_3^2 + 4608x_1^4x_3^4 + 4608x_1^4x_2^4 \\
&+ 4608x_1^4x_4^4 + 4608x_2^4x_4^4 + 4608x_2^4x_3^4 - 9216x_1^2x_2^4x_3^2 \\
&+ 9216x_1^2x_2^2x_3^4 - 9216x_1^2x_3^2x_4^4 + 9216x_1^4x_3^2x_4^2 \\
&- 9216x_1^2x_3^4x_4^2 - 9216x_1^4x_2^2x_3^2 + 9216x_1^2x_2^2x_4^4 \\
&- 9216x_1^4x_2^2x_4^2 - 9216x_1^2x_2^4x_4^2 + 768x_1^8 + 768x_2^8 \\
&+ 768x_3^8 + 18432x_2^2x_1^2x_4^2x_3^2
\end{aligned} \tag{70}$$

It is not difficult to show that the Ricci scalar curvature takes the intriguing form

$$R = -\frac{9}{2} \frac{x_1^2 - x_3^2 - x_4^2 + x_2^2}{(b + 3x_1^2 - 3x_3^2 + 3x_2^2 - 3x_4^2)} \frac{1}{(b^2 + r_{m1}b + r_{m0})} \tag{71}$$

where the moduli functions r_{m1} and r_{m0} can be defined as

$$\begin{aligned}
r_{m1} &:= (4x_1^2 + 4x_2^2 - 4x_3^2 - 4x_4^2) \\
r_{m0} &:= (+3x_1^4 + 3x_4^4 - 6x_1^2x_3^2 - 6x_1^2x_4^2 + 6x_1^2x_2^2 \\
&+ 6x_3^2x_4^2 - 6x_2^2x_3^2 - 6x_2^2x_4^2 + 3x_2^4 + 3x_3^4)
\end{aligned} \tag{72}$$

For a pairwise equality: $x_1 = x_2 = x$, $x_3 = x_4 = y$ and with the unit Fayet parameter $b = 1$, the principle minors are given by

$$g_2 = -256x^2y^2 + 192x^4 + 16 + 128x^2 - 64y^2 + 64y^4 \tag{73}$$

$$\begin{aligned}
g_3 &= -64 - 1792x^4 - 1280y^4 + 3072x^2y^2 - 640x^2 + 512y^2 \\
&+ 4096x^4y^2 - 3584y^4x^2 + 1024y^6 - 1536x^6
\end{aligned} \tag{74}$$

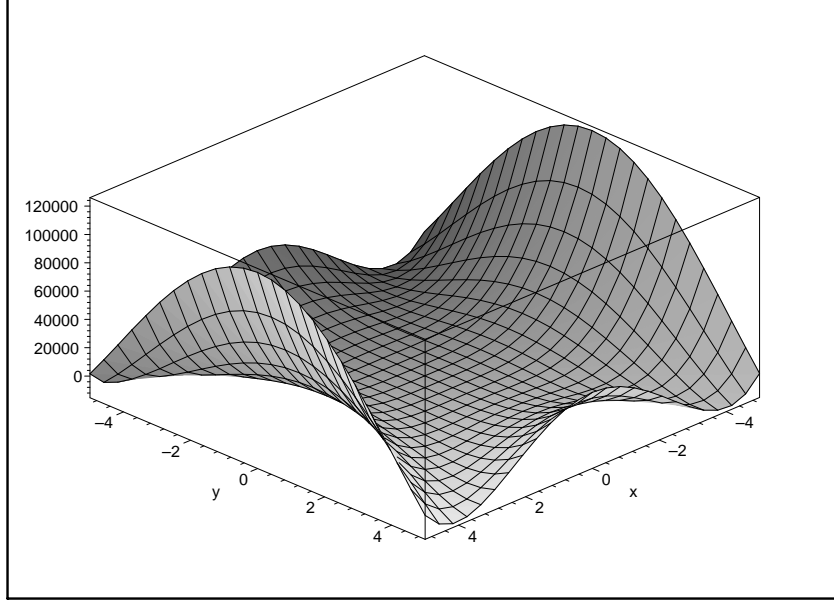


Figure 11: The surface minor plotted as a function of the moduli fields, x, y , describing the fluctuations in the threshold configuration.

$$\begin{aligned}
g = & 256 + 12288x^4 + 12288y^4 - 24576x^2y^2 + 3072x^2 \\
& - 3072y^2 - 61440x^4y^2 + 73728x^4y^4 - 49152x^6y^2 \\
& - 49152y^6x^2 + 61440y^4x^2 + 12288x^8 + 12288y^8 \\
& - 20480y^6 + 20480x^6
\end{aligned} \tag{75}$$

In this case, the associated scalar curvature is given by

$$R = -9 \frac{(x^2 - y^2)}{(1 + 6x^2 - 6y^2)(1 + 12x^4 + 8x^2 - 24x^2y^2 + 12y^4 - 8y^2)} \tag{76}$$

The behavior of the principle minors g_2, g_3 , the determinant of the metric tensor g and the corresponding scalar curvature R have been respectively shown in the Figs.(11, 12, 13, 14).

For equal values of the scalar fields, viz, $x = z, y = z$ the respective limiting values of the principle minors and determinant of the metric tensor are given by

$$g_2 := 16b(b + 4x^2) \tag{77}$$

$$g_3 := -64b^2(b + 2x^2) \tag{78}$$

$$g := 256b^4 \tag{79}$$

In this case, we find that the associated scalar curvature vanishes identically, viz., for all values of the Fayet parameter, we have a non-interacting statistical configuration with the following scalar curvature

$$R = 0 \tag{80}$$

Interestingly, we note, for the limiting threshold configuration with the vanishing Fayet parameter $b = 0$, that the stabilities on a line, surface, hyper-surface, as

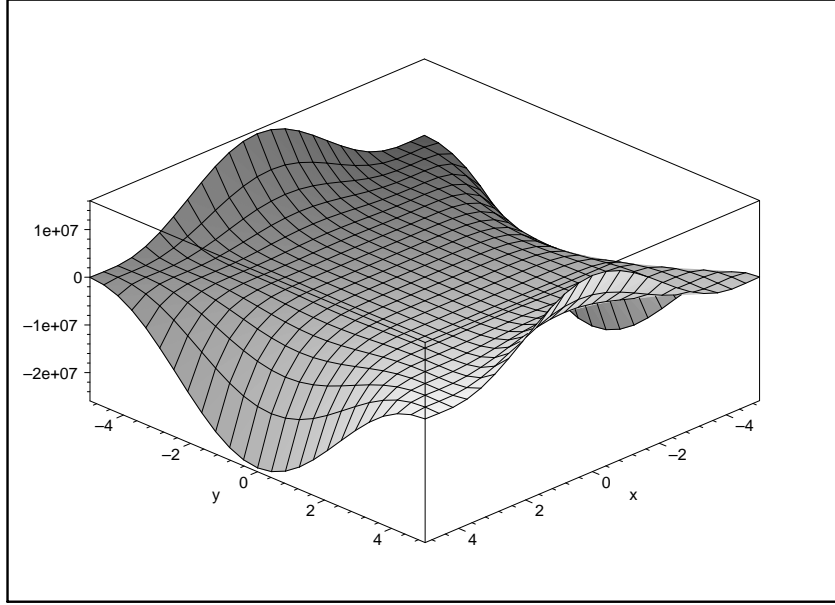


Figure 12: The hypersurface minor plotted as a function of the moduli fields, x, y , describing the fluctuations in the threshold configuration.

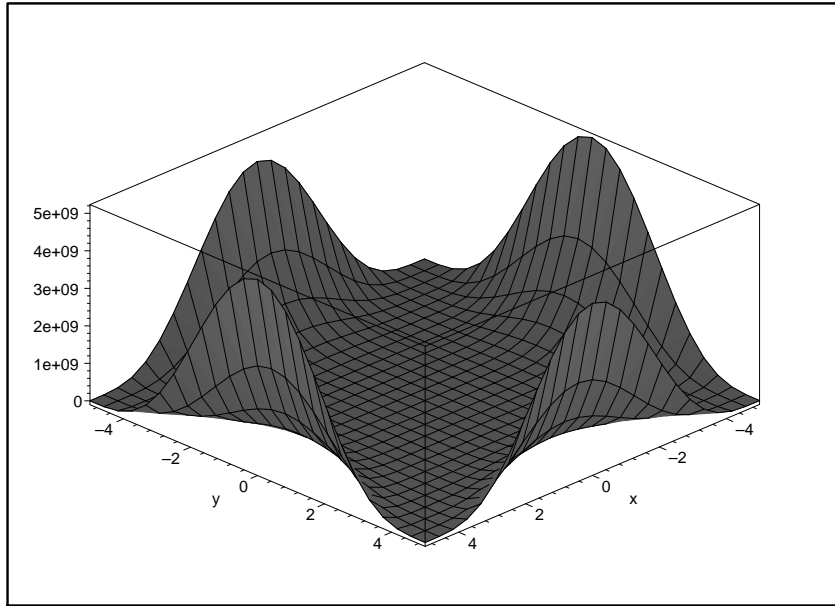


Figure 13: The determinant of the metric tensor plotted as a function of the moduli fields, x, y , describing the fluctuations in the threshold configuration.

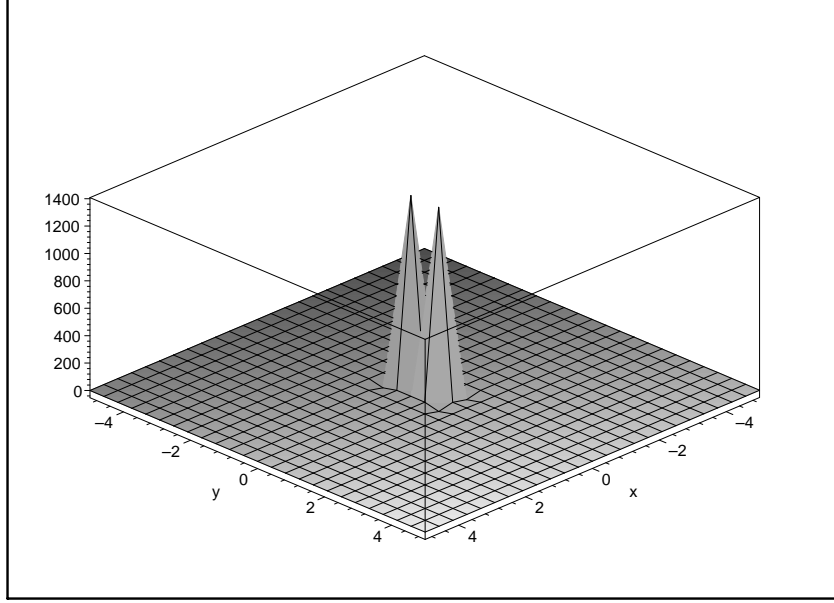


Figure 14: The scalar curvature plotted as a function of the moduli fields, x, y , describing the fluctuations in the threshold configuration.

well as the stability of the entire moduli configuration, are respectively described by the properties of polynomials of degree 2, 4, 6 and 8. Specifically, for the limiting Fayet parameter $b = 0$, we find, from the perspective of the intrinsic geometry, that the diagonal and off-diagonal pair correlations reduce to a set of quadratic polynomials and monomials. For example, the first diagonal and the first off-diagonal components reduce to the following expressions

$$\begin{aligned} g_{x_1 x_1} &= 12x_1^2 + 4x_2^2 - 4x_3^2 - 4x_4^2 \\ g_{x_1 x_2} &= 8x_1 x_2 \end{aligned} \quad (81)$$

The surface minor reduces to the following quartic polynomial

$$\begin{aligned} g_2 &= -64x_1^2 x_3^2 + 96x_2^2 x_1^2 + 16x_4^4 - 64x_2^2 x_4^2 + 48x_2^4 \\ &\quad + 16x_3^4 + 32x_3^2 x_4^2 + 48x_1^4 - 64x_2^2 x_3^2 - 64x_1^2 x_4^2 \end{aligned} \quad (82)$$

The question of the stability of the hypersurface of the moduli configuration may be determined by the associated minor of the metric tensor. It follows that the hypersurface minor is given by

$$\begin{aligned} g_3 &= 896x_1^2 x_2^2 x_4^2 - 576x_1^4 x_2^2 + 448x_1^4 x_4^2 - 576x_1^2 x_2^4 \\ &\quad + 1152x_1^2 x_2^2 x_3^2 - 896x_1^2 x_3^2 x_4^2 - 192x_2^6 + 64x_4^6 \\ &\quad + 192x_3^6 - 192x_1^6 - 320x_1^2 x_4^4 - 320x_2^2 x_4^4 + 576x_1^4 x_3^2 \\ &\quad + 448x_2^4 x_4^2 + 320x_3^2 x_4^4 - 576x_1^2 x_3^4 + 576x_2^4 x_3^2 \\ &\quad - 576x_2^2 x_3^4 - 896x_2^2 x_3^2 x_4^2 + 448x_3^4 x_4^2 \end{aligned} \quad (83)$$

Finally, an existence of the global stability of the vanishing Fayet parameter threshold moduli configuration may be determined by the positivity of the determinant of the underlying metric tensor. In this limit, we find that the determinant of the metric tensor takes the following homogeneous polynomial form

$$g = 9216x_1^4 x_3^2 x_4^2 + 9216x_2^4 x_4^2 x_3^2 + 9216x_1^2 x_2^2 x_4^4$$

$$\begin{aligned}
& +9216x_1^2x_2^2x_3^4 + 18432x_2^2x_1^2x_4^2x_3^2 + 4608x_3^4x_4^4 \\
& +3072x_3^6x_4^2 - 3072x_3^6x_1^2 - 3072x_3^6x_2^2 - 3072x_1^6x_4^2 \\
& -3072x_1^6x_3^2 + 3072x_2^6x_1^2 - 3072x_2^6x_3^2 - 3072x_4^6x_2^2 \\
& +3072x_4^6x_3^2 + 768x_3^8 + 768x_1^8 + 768x_2^8 + 768x_4^8 \\
& +4608x_1^4x_3^4 + 4608x_1^4x_4^4 + 4608x_2^4x_4^4 + 4608x_2^4x_3^4 \\
& +4608x_1^4x_2^4 - 9216x_1^2x_2^4x_3^2 - 9216x_1^4x_2^2x_3^2 \\
& -9216x_2^2x_4^4x_3^2 - 9216x_1^2x_2^4x_4^2 - 9216x_1^2x_3^2x_4^4 \\
& -9216x_2^2x_3^4x_4^2 - 9216x_1^4x_2^2x_4^2 - 9216x_1^2x_3^4x_4^2 \\
& +3072x_1^6x_2^2 - 3072x_2^6x_4^2 - 3072x_4^6x_1^2
\end{aligned} \tag{84}$$

Moreover, in the limit of $b = 0$, we observe that the Ricci scalar curvature reduces to the inverse square of a unique quadratic polynomial of the moduli scalars, which defines the D -term potential of the associated threshold configuration. Specifically, we find that the limiting scalar curvature may be expressed as

$$R = -\frac{1}{2} \frac{1}{(x_1^2 + x_2^2 - x_3^2 - x_4^2)^2} \tag{85}$$

For the case of the reflected Fayet parameter $b = -a$, the components of the metric tensor transform as per our general expectation. For example, the first diagonal and the first off-diagonal components of the metric tensor transform as follows

$$\begin{aligned}
g_{x_1x_1} &= 12x_1^2 + 4x_2^2 - 4x_3^2 - 4x_4^2 - 4a \\
g_{x_1x_2} &= 8x_1x_2,
\end{aligned} \tag{86}$$

Under the reflection of the Fayet parameter $b = -a$, we find that the following value is satisfied by the planar minor,

$$g_2 := 16a^2 - g_{21}a + g_{20} \tag{87}$$

Nevertheless, it is not difficult to see that the hyper-planar minor reduces to the following expression

$$g_3 := -64a^3 + g_{32}a^2 - g_{31}a + g_{30} \tag{88}$$

Under the reflection of the Fayet parameter, the following expression gives an intriguing comparison of the determinant of the metric tensor

$$g := g_{44}a^4 - g_{43}a^3 + g_{42}a^2 - g_{41}a + g_{40} \tag{89}$$

Similarly, one may easily check that the scalar curvature satisfies,

$$R := \frac{9}{2} \frac{(x_1^2 - x_3^2 - x_4^2 + x_2^2)}{(a - (3x_1^2 + 3x_3^2 - 3x_2^2 + 3x_4^2))} \frac{1}{(a^2 - r_{m1}a + r_{m0})} \tag{90}$$

Under the inverse of the Fayet parameter $b := 1/c$, we observe that only the diagonal pair correlations dilate, while the off-diagonal pair correlations remain intact. For instance, the first diagonal and the first off-diagonal components of the metric tensor scale as follows

$$\begin{aligned}
g_{x_1x_1} &= \frac{4}{c}((3x_1^2 + x_2^2 - x_3^2 - x_4^2)c + 1) \\
g_{x_1x_2} &= 8x_1x_2
\end{aligned} \tag{91}$$

In this case, we see that the series of the polynomials, which define the associated principle minors, get inverted in the Fayet parameter. Under the inversion $b = 1/c$, we may further observe that the stability of the configuration can stem from the following transformation rules. In particular, we find that the respective stability constraints transform as

$$g_2 = \frac{1}{c^2}(g_{22} + g_{21}c + g_{20}c^2) \quad (92)$$

$$g_3 = \frac{1}{c^3}(g_{33} + g_{32}c + g_{31}c^2 + g_{30}c^3) \quad (93)$$

$$g = \frac{1}{c^4}(g_{44} + g_{43}c + g_{42}c^2 + g_{41}c^3 + g_{40}c^4) \quad (94)$$

Thus, we find that the stability constraints remain alternating and scale by the inverse of the Fayet parameter, under its inversion. A straightforward calculation shows that a similar scaling property arises for the intrinsic scalar curvature. In this case, we see that the scalar curvature transforms into the following expression

$$R = -\frac{9}{2}c^3 \frac{x_1^2 - x_3^2 - x_4^2 + x_2^2}{(1 + 3c(x_1^2 - x_3^2 + x_2^2 - x_4^2))} \frac{1}{(r_{m0}c^2 + r_{m1}c + 1)}, \quad (95)$$

where the coefficients r_{m0} and r_{m1} remain intact as per their earlier definition. For equal values of the moduli scalars, the planar, hyper-planar and the determinant stability constraints imply that the principle minors and the determinant of the metric tensor respectively transform as $g_2 = (8x^2 + 4/c)^2 - 64x^4$, $g_3 = -64(1 + 2x^2c)/c^3$ and $g = 256/c^4$. For a given nonzero c , this shows that the equality of the constituent moduli fields implies a relatively unstable intrinsic hypersurface, pertaining to the threshold configurations.

We thus see that the determinant of the metric tensor is positive definite. This suggests that the entire threshold moduli configuration may be stabilized, when all the constituent fields are allowed to fluctuate. Furthermore, one may easily notice that the present investigation provides an existence of the intriguing genesis of moduli stabilization with a vanishing intrinsic scalar curvature, which specifically occurs for equal vacuum expectation values of the constituent moduli scalars. Similar intrinsic geometric studies may further be investigated, in order to acquire the stability properties of the moduli configurations.

5 Conclusion and Remarks

The present paper explores the intrinsic geometric stability properties for the moduli configurations. We exhibit an interesting set of implications for the D -term potentials. In this study, we find that there exists a set of real intrinsic Riemannian geometric stability criteria and thereby provide the underlying implications for the Fayet models and walls of the marginal and threshold stability properties pertaining to the D -term moduli configurations. More specifically, we find, for a set of given complex abelian scalar fields satisfying the D-constraint, that the various possible stability criteria, viz., linear, planar, hyper-planar and the entire moduli space stabilities, may all be easily accomplished over a domain of the Fayet parameter. The present paper analyzes the moduli stabilization problem for the marginal and threshold stable moduli configurations and thereby explicates that these configurations are rather stable, under the possible transformations of the Fayet parameter.

We supply an intrinsic geometric method to investigate the nature of the marginal and threshold walls of the stabilities. Furthermore, we have demonstrated that the

threshold stable configurations concentrate on the alternating definition of the D -term potential. Thus, the examination of the underlying intrinsic configuration depends on the true symmetry of the underlying moduli potential. Nevertheless, the non-vanishing of the intrinsic scalar curvature defines the possible range of the global correlations between the constituent real scalar fields. Whereas, the vanishing curvature indicates that the underlying moduli space configuration comes out to be a non-interacting statistical system. After the vanishing value of the scalar curvature, the configuration changes the nature of the underlying global interaction among the scalar fields.

More precisely, the intrinsic scalar curvature defines an expected volume of the global correlation among the constituent scalar fields, beyond which the underlying moduli configuration becomes a non-interacting system and thereby changes its intrinsic geometric nature. The singularity structure of the intrinsic curvature determines possible phase transitions, if any, in the vacuum moduli manifold, and thereby infers about its feasible decay to the different daughter configurations. It is worth to mention that the threshold stable configurations do not always entail the positivity of the principle components of the metric tensor, higher principle minors and the determinant of the intrinsic metric tensor. Thus, there does not always exist a well-defined volume form on the underlying vacuum manifold of the threshold moduli configurations. This supports an intriguing fact that the marginally stable configurations swiftly decay to another set of non-supersymmetric BPS-configurations.

We further find that the marginally stable BPS configurations have the same sign of the principle components of the metric tensor. In turn, it follows that the two and four dimensional real intrinsic configurations have a uniform sign for the all components and are positive definite. Thus, the above moduli configurations are stable for all non-negative values of the Fayet parameter. In general, we notice that very similar conclusions are noticed for both the leading order potentials describing decaying BPS configurations. In particular, we have shown that the marginal configurations, with the unit $U(1)$ charge $q_i = 1, \forall i$, possess the same intrinsic geometric metric tensor. The nature of principle minors and scalar curvature have also an expected promising feature. The general properties of the above geometric quantities remain alternating, for the threshold configurations defined with the charges $q_i = 1, q_{i+1} = -1 \forall i$. As mentioned before, such moduli configurations may be piecewise analyzed. From the perspective of the intrinsic geometry, we envisage that the present analysis may be extended for a variety of moduli configurations, e.g., D -term and F -terms potentials.

Importantly, the subleading corrections nevertheless need be carefully explored, in order to have a more precise evaluation of the symmetries of the general moduli configurations, which are entangled with the invariants of the intrinsic geometry. In particular, it may be important to distinguish whether the underlying reflection symmetry and possible scalings are preserved, under the higher derivative contributions. This task is left for a future investigation. As mentioned earlier, it is worth mentioning that a moduli configuration is said to be stable, if the basic intrinsic geometric elements, *viz.*, the diagonal components of the metric tensor g_{ii} , principle minors g_i and the determinant of the metric tensor g are positive definite $\forall i \in \Lambda$, constituting the stability in each possible direction of the chosen moduli configuration. For a finite range of the Fayet parameter, the framework of the present approach supports that the general moduli configurations may be stabilized in a finite domain of the moduli fields.

We may further observe that the principle components of the metric tensor retain the same polynomial form in the constituent scalar fields, while the off-diagonal components turn out to be some quadratic monomials. The present investigation shows that the definite character of the intrinsic geometry remains true, for an arbitrary number of scalar fields satisfying the leading order D -term potential. For the

threshold moduli configurations, the intrinsic scalar curvature, thus investigated, vanishes identically for the vanishing value of the Fayet parameter. Whilst, it is not difficult to show that similar facts do not continue to hold for the underlying determinant of metric tensors. In particular, we discover that the null value of the Fayet parameter does not cause the determinants to vanish. Consequently, our computation demonstrates that the marginally stable configurations are intrinsic geometrically curved, even for the vanishing value of the Fayet parameter, whilst the same conclusion does not hold for threshold stable moduli configurations. In other words, it may be generically anticipated, for the threshold moduli configurations, that the intrinsic scalar curvature vanishes identically for the limiting Fayet parameter $b = 0$. Thus, the threshold configurations become a non-interacting statistical system, without the contribution of the Fayet term.

Furthermore, faithfully equivalent conclusions remain true in common, for equal vacuum values of the scalar fields. In particular, we find that the marginally stable moduli configurations are interacting with an underlying intrinsic curvature $R(x_i) \neq 0$, for identical values of the constituent scalars $x_i = x$. Whilst, we find for the vanishing Fayet parameter that the corresponding scalar curvature $R(x_i)$ vanishes identically for the threshold stable configurations, pertaining to the D -term potential and the associated Fayet model. An inference may thus be made that the marginal configurations do not change the nature of the intrinsic geometry, under an increment of vacuum scalar fields, whereas the threshold configurations do change their intrinsic geometric characteristics. The present analysis thus offers a stimulating feature for the possible decaying supergravity configurations. Specifically, the precise notions, as obtained in the present case, may shed light on the symmetry properties of the intrinsic geometric invariant arising from the generic D -term and other moduli space potentials.

Finally, the divergence structure of the intrinsic curvature implies possible phase transitions, in the chosen physical vacuum moduli configurations. As per the guidelines of the present consideration, such a characterization may be furnished by a finite set of abelian scalar fields. Although the present analysis is of leading order in character, however it provides a general real intrinsic geometric covariant technology, which could be applied towards stability examinations of the various moduli configurations, pertaining to the gauge and string theories. In particular, it is worth mentioning that the moduli space stabilization problem may be appreciated, from the outset of the intrinsic real Riemannian geometry. In this perspective, the potential application may be explored for the determination of local and global moduli stability domains, involving a class of moduli configurations of the present experimental and phenomenological interests of the subject. As mentioned in the introduction, some of these include Higgs moduli, Calabi-Yau moduli, torus moduli, instanton moduli, topological string moduli, and the moduli pertaining to D and M -particles. In order to encompass an explicit stability feature emerging from the present setup, the concept of the intrinsic geometry may herewith be brought out into a close connection with the underlying supergravity configurations. From the perspective of the present investigation, it may therefore be anticipated that a set of similar observations would emerge further, for a class of generic moduli space configurations. This problem is left for a future consideration.

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